Calculation of the transition matrix and of the occupation probabilities for the states of the Oslo sandpile model

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The Oslo sandpile model, or if one wants to be precise, ricepile model, is a cellular automaton designed to model experiments on granular piles displaying self-organized criticality. We present an analytic treatment that allows the calculation of the transition probabilities between the different configurations of the system; from here, using the theory of Markov chains, we can obtain the stationary occupation distribution, which tells us that the phase space is occupied with probabilities that vary in many orders of magnitude from one state to another. Our results show how the complexity of this simple model is built as the number of elements increases, and allow, for small system sizes, the exact calculation of the avalanche-size distribution and other properties related to the profile of the pile.

I. INTRODUCTION AND DEFINITION OF THE OSLO MODEL

Self-organized criticality (SOC), born from the deep insights of Bak *et al.*, deals with the emergence of scale invariance in slowly driven nonequilibrium systems [1,2]. The phenomenon is illustrated with the archetypical example of a pile of sand, and realized in computer simulations of diverse sandpile models, which are mainly based on the original Bak-Tang-Wiesenfeld (BTW) model [3,4]. However, the relevance of SOC for real granular matter was unclear until Frette *et al.* [5] performed experiments on a $1+1$-dimensional pile of rice; these experiments and some others [6–8] were modeled with a cellular automaton introduced in Ref. [6], later called the Oslo model.

The Oslo model has the interest of being (as far as I know) the first SOC sandpile model or, more appropriately, ricepile model, able to reproduce experimental results. For the avalanche properties [5], the concordance with experiments is only qualitative, whereas for the transport of individual grains [6], and for the surface roughness [7], the agreement is also quantitative. Moreover, the Oslo model is remarkable as a simple model of SOC, because it displays this nontrivial behavior in one dimension.

The model is designed to mimic the experimental situation in Refs. [5–7]: grains are slowly added at a fixed position on a quasi-one-dimensional substrate which is in between two parallel vertical plates; just at the (let us say) left of the position of addition a wall prevents the falling of the grains; on the other side, the right boundary is open. The model assumes a discrete space, $x=1,2,\ldots,L$, from left to right, as well as discrete time and field (the height of the pile, or number of grains). The grains pile up in columns until the local slope somewhere is too large, then the upper grain becomes unstable and is transferred to the next column to the right (from x to $x+1$). This transfer can induce further instabilities and therefore a chain reaction or avalanche. A

slow-driving condition imposes that during the avalanches the external addition of grains is interrupted; this implies that the evolution of the model (and of the experiments) takes place in two separate time scales: a slow time scale for the grain addition and a fast time scale for the evolution of the avalanches.

In terms of the height of the x column, $h(x)$, and the local slope, defined as $z(x)\equiv h(x)-h(x+1)$, taking $h(L+1)\equiv 0$, these prescriptions are expressed in the following rules [6]:

$$\text{if } z(x)\leqslant z_{th}(x)\forall x\Rightarrow h(1)\rightarrow h(1)+1,$$

$$\text{if } \quad z(x)>z_{th}(x)\Rightarrow \begin{cases} h(x)\rightarrow h(x)-1 \\ h(x+1)\rightarrow h(x+1)+1 \\ z_{th}(x)\rightarrow n_{rand} \end{cases} \quad (1)$$

(where the update is supposed to take place in parallel). Here $z_{th}(x)$ refers to a local threshold, which rather than being constant changes with every toppling at x to a random value n_{rand}, chosen as

$$n_{rand}=\begin{cases} 1 & \text{with probability } p \\ 2 & \text{with probability } q=1-p. \end{cases} \quad (2)$$

These simple rules, Eqs. (1) and (2), together with the boundary condition, $h(L+1)\equiv 0$, completely define the Oslo model. Nevertheless, it is convenient to express rule (1) in terms only of the slope, turning out to be

$$\text{if } z(x)\leqslant z_{th}(x)\forall x\Rightarrow z(1)\rightarrow z(1)+1,$$

$$\text{if } z(x)>z_{th}(x)\Rightarrow \begin{cases} z(x)\rightarrow z(x)-2 \\ z(x-1)\rightarrow z(x-1)+1 \\ z(x+1)\rightarrow z(x+1)+1 \\ z_{th}(x)\rightarrow n_{rand} \end{cases} \quad (3)$$

for $x=1,2,\ldots,L-1$, and

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$$\text{if } z(L) > z_{th}(L) \Rightarrow \begin{cases} z(L) \rightarrow z(L) - 1 \\ z(L-1) \rightarrow z(L-1) + 1 \\ z_{th}(L) \rightarrow n_{rand} \end{cases}$$

(taking into account that at $x = 0$ the z variable is not defined). Notice how the dynamics of the grains has allowed to define a different dynamics for the units of slope, which can be considered as some kind of virtual particles. Both dynamics are conservative, except at the boundaries, which curiously are reversed: the open end for the grains at $x = L$ is a closed boundary for the slopes and vice versa at $x = 1$. It is important to have this in mind to avoid confusion.

The Oslo model is essentially the one-dimensional BTW model, but with fixed addition at $x = 1$ and an open boundary condition for the grains at $x = L$. The key different ingredient, which makes the model critical in one dimension, is the selection of dynamically changing thresholds, to account for the heterogeneities of a real system. In this way, whereas the randomness in the BTW sandpile is external, in the Oslo model it is internal, as in real ricepile experiments. This spirit is original from the philosophy of Ref. [9], although the model there is much more complicated.

Even with the simplicity of its definition, Eqs. (1) and (2), the Oslo model gives rise to an astonishing complex behavior. As is usual in SOC systems, it shows a power-law distribution of avalanche sizes [6,10] and avalanche durations [10], signaling the existence of no characteristic scales for the avalanche process. But also it has been shown that the transport of the grains through the pile is anomalous, in the sense that there is no normal diffusion but a power-law distributed transit time [6], spanning many orders of magnitude (just as it happens in the experiments, as we have mentioned). This has been explained by the fact that the time that a grain is trapped at a given position is also broadly distributed [11,12], which has in its turn been related to a kind of skewed fractional Brownian motion for the variations of the height, in the antipersistent case [13]. Moreover, the distances traveled by the grains during an avalanche turn out to be Lévy flights [11,12], i.e., again scale-free, despite the nearest-neighbor rules. Additionally, the time fluctuations of the profile scale with a roughness exponent that is in good concordance with the experiments [6,7]. In fact, the exponents of all previous magnitudes can be related by several scaling laws.

Further, the time sequence of the transit times shows a clear multifractal spectrum [14], whereas the time sequence of the mean slope displays $1/f$ noise [15], in contrast to the BTW model [2]. The model also allows one to study the transition from intermittent behavior to continuous flow, just increasing the driving rate and breaking the time-scale separation [16]. Very recently it has been shown that a small damage performed in the system does not spread, and therefore the sensitivity of the model to the initial conditions is quite different to that of chaotic systems and to what was expected from systems "at the edge of chaos" [17]. On the other hand, there exists an exact mapping between this model an a model of interface depinning, which establishes the existence of a wide universality class for these nonequi-

librium systems related to the quenched Edwards-Wilkinson equation [18–21]. In a previous paper, I have also signaled the similarities between the Oslo model and the recurrence of real earthquakes [22]. Also, variations and extensions of the basic idea have been considered [23–29]. Because of its simple definition and rich phenomenology related to experimentally observable quantities, we can consider the Oslo model as the analog of the Ising model for slowly driven complex systems [30].

In spite of these fascinating properties, our understanding of the Oslo model comes mainly from computer simulations and some scaling arguments; no analytical solution exists or seems possible in the near future (as is the general case for nonequilibrium systems). Hence, the exact enumeration of the number of states in the attractor for this model, performed by Chua and Christensen, is very remarkable [31]. They found that this number increases exponentially with system size as

$$N_A = \frac{aG^L + a^{-1}G^{-L}}{\sqrt{5}}, \tag{4}$$

with G the golden mean, $G = (3 + \sqrt{5})/2 \simeq 2.6$, and $a = (1 + \sqrt{5})/2 \simeq 1.6$.

In this paper, we are going to derive exact expressions for the transition probabilities between states in the Oslo model; using the results for a system of size $L - 1$ we will get these probabilities in a system of size L. The increase in just one unit of the size of the system leads to an enormous increase in the complexity of the resulting equations; we have a kind of machine for building complexity. With the transition probabilities it is possible to obtain the stationary occupation distribution, which is the probability with which a state is visited in the asymptotic regime. We will get that, in contrast to other SOC models, the states in the attractor are not equally likely; rather, the range of occupation probabilities varies dramatically in many orders of magnitude. Once the stationary occupation distribution is known, several other quantities, directly related with the profile of the pile, as the mean-slope distribution and the avalanche-size distribution, can be obtained. (I have very recently become aware that Dhar has undertaken an analysis of precisely the same problem [32]; nevertheless, his approach is entirely different from ours and both works can be considered as complementary to each other.)

II. SOME PROPERTIES OF THE MODEL

Two types of states, or configurations, are possible in the system: unstable states, with at least one local slope value above its local threshold, $z(x) > z_{th}(x)$, and metastable states, where all the slopes are below threshold, $z(x) \leq z_{th}(x) \forall x$. Unstable states evolve by means of avalanches towards metastable states, but the addition of a new grain can make metastable configurations become unstable again, and so on.

We are only interested on metastable states. If we assume that the initial slopes $z(x)$ are not negative [33], the possible stable values for this variable are 0, 1, and 2 (values ≥ 3 are

always unstable); therefore, in a system of size L a metastable configuration will be fully specified by an L vector whose components are the slope values 0, 1, or 2. For instance, we can consider a state s to be $s=\{210\ldots1\}$ which means $z(1)=2$, $z(2)=1$, $z(3)=0$, and so on up to $z(L)=1$ (alternatively, the metastable state s can be viewed as an integer with L digits expressed in base 3). Notice that we do not need to keep track of the thresholds $z_{th}(x)$; this is so because the dynamics can be described in an alternative but equivalent way, using the following rule:

$$\text{if } z(x)\text{has just} \atop \text{changed its value to} \left\{ \begin{array}{l} 1 \text{ or less, no toppling} \\ 2, \text{ toppling with probability } p \\ 3 \text{ or greater, toppling.} \end{array} \right. \tag{5}$$

We stress that this rule has to be applied only when the site at x has received one (or more) units of slope or has toppled in the previous avalanche (fast) time step. (This works because we have only defined two possible values for the threshold; with three values the situation would be more delicate.)

A very useful property in order to study the evolution of the system will be the Abelian symmetry, first considered in sandpile models by Dhar [34]. It states that the order in which units of slope are added and sites over threshold topple does not matter for the final configuration of the pile; therefore, we will be allowed to topple the sites in the most convenient sequence to keep the process manageable in the calculations. The demonstration of this property in our case is similar to that in Refs. [34–36] but takes into account that we have evolving thresholds. If we consider two sites x and y that are unstable it is easy to see that we get the same state no matter which one topples first, since after the toppling of x, site y will still remain unstable, and the quantity that any toppling site is reset and the quantity transferred to the neighbors will be the same, independently of the order. The same reasoning can be extended to more than two overthreshold sites. But this is so only if the random thresholds for sites x and y are equally chosen in each possible sequence of topplings, that is, we need a predefined sequence of thresholds at each site, or, from a computational point of view, instead of having a single random number generator, a different generator must be used for each site. From a similar reasoning as before, the addition of grains (or slope) at $x=1$ commutes with the toppling of any unstable site.

On the other hand, the evolution of the pile can be described by means of a finite Markov chain; indeed, the probability that a given state transforms into another state depends only on the two states, and not on the previous history, thanks to rule (5). In particular, the probability that a metastable state i evolves to a new metastable state j after the addition of one grain and at the end of the corresponding avalanche is independent of the previous states of the pile, and can be obtained by means of the unstable states that separate the states i and j. These probabilities constitute a matrix that will be referred to as $\mathcal{W}$, with elements W_{ij}. Probability theory imposes that $W_{ij}\geq0$ and that the files of the matrix are normalized to 1 (in the probabilistic sense), i.e., $\Sigma_j W_{ij}=1$. A matrix with these properties constitutes a stochastic matrix. (Further, as $\mathcal{W}$ is constant through the time evolution, we are dealing with a homogeneous Markov chain.)

III. CHARACTERIZATION OF THE ATTRACTOR AND EXISTENCE OF A UNIQUE STATIONARY DISTRIBUTION

Some results from the theory of Markov chains can be applied at this point. To get the stationary properties of the pile it will be crucial to have a well defined stationary occupation distribution; this quantity gives the probability with which every state is visited in the asymptotic regime, that is, in the attractor, and it is represented by a vector where each component corresponds to a configuration of the system. The stationary occupation distribution is simply referred to as the stationary distribution in Markov-chain theory, but here we are interested in many other probability distributions in the stationary case, as for instance that for the avalanche sizes. (Another common name is ergodic distribution.)

For completeness, let us explain that the attractor can be defined as the set of recurrent, or persistent states, these being the states for which the return probability is exactly 1. More precisely, if a state is visited at some time, there is a probability 1 that it will be visited again in the future. In contrast, for transient states this probability is smaller than 1, or even zero.

At this point it is convenient to use graph theory to represent a Markov chain: the transition probability matrix $\mathcal{W}$ defines a graph where nodes i and j are directly connected if $W_{ij}\neq0$, otherwise, there is no direct link between i and j; that is, we have the graph of the possible transitions in one (slow) time step.

The existence of a unique stationary occupation distribution, independent of the initial conditions, is guaranteed if the graph associated with the matrix $\mathcal{W}$ has only one nonperiodic final class [37] (this is also a necessary condition). A final class is a strongly connected component whose elements have no transitions to elements outside the class. A strongly connected component is a part of the graph in which any pair of nodes, or states, can be connected in both directions; in other words, i and f stay in a circuit and it is possible to reach state f starting from i and vice versa. A final class represents then nothing else than an attractor in which the system can settle after a transient period. The periodicity of a strongly connected component is the greatest common divisor of the length of their circuits; if this number is 1 the graph is nonperiodic. This is, for instance, the case in the presence of loops (circuits with just one element, that is, $W_{ii}\neq0$, for some i).

Let us see which states of the pile constitute the attractor, or final class. We have found it simpler to consider the connections between two states by means of the steepest metastable state, which is $\{22\ldots2\}$ [i.e., the one with $z(x)=2\forall x$], and then show which states lead to the steepest state and which ones result from it.

In fact, all states can lead to the steepest state. To show this, we add grains and let the sites topple depending on their local thresholds, but after every toppling we assume that the

maximum threshold $z_{th}(x)=2$ is always selected; then, we are essentially in the same situation as in the one-dimensional BTW model: every column in the pile grows to reach the steepest profile. In this way, adding enough grains we will end in the steepest state. This is more easily seen applying the Abelian symmetry: we first add grains to build the first column, until it reaches the desired height, $h(1)=2L$, then we add more grains and let them topple to the second column until it reaches $h(2)=2(L-1)$, and so on.

This is enough to ensure that there is only one final class, although the only thing we know about it is that it includes the steepest state. If we continue with the characterization of the final class we will simply get the attractor studied in Ref. [31].

Contrary to the previous situation, not every state is reachable from the steepest state. Consider first final states without zeros, i.e., $z_f(x)=1$ or 2, only $\forall x$. One way to get these states is the following: after the addition of the first grain, which crosses the whole pile arriving to the exit, we fix the thresholds to the slopes of the desired final configuration, $z_{th}(x)=z_f(x)$ $\forall x$. Then, we apply the Abelian property and start the toppling process of the remaining grains from the rightmost site $x=L$, emptying out this column [from $z(L)=2$ to $z(L)=z_f(L)$]; after this, we take the next column to the left, $L-1$, and let every extra grain topple until it leaves the pile. Repeating the same procedure we end in the desired state (which is therefore reachable after just one avalanche). Basically, we are in a one-dimensional BTW-like situation again, where the pile tends to a state $z(x)=z_{th}(x)$.

When there are zeros in the configuration, $z(x)=0$ for some x, we cannot apply this trick since thresholds are defined as larger than zero. In fact, Chua and Christensen [31] have noticed that these states do not necessarily belong to the attractor: they argue that zeros have to be compensated by twos [i.e., sites with slope $z(x)=2$]; to be precise, they show that a necessary condition to belong to the attractor is that for each zero-slope site in the configuration there must be at least one two-slope site to the right, before the next zero or before the exit.

Let us see that Chua and Christensen's condition is also sufficient to belong to the attractor: starting from the steepest state, the first zero appears when (after a number of topplings) a site with $z(x)=2$ topples [if $z_{th}(x)=1$] and the following site has $z(x+1)=1$. Application of the rules gives then $z(x)=0$ and $z(x+1)=2$; if $z_{th}(x+1)=2$ this site does not topple and the configuration can be metastable. Now that a zero exists, the same process can be repeated but with the zero-slope site receiving a grain; that is, we can have $z(x-1)=2$ and $z(x)=0$, if $z_{th}(x-1)=1$ we get $z(x-1)=0$, $z(x)=1$, and $z(x+1)=2$; this means that the zero can move to the left, but is somehow associated with the existence of a site with slope 2, and this slope 2 cannot disappear if the zero exists. In order to topple, site $x+1$ would need the addition of one grain, but grains come from the left and cannot reach $x+1$ except if the zero is removed, i.e., any grain coming from the left would encounter the zero slope and by the rules of the model would stick there.

In general, to get a configuration $\{\ldots 01 \ldots 12 \ldots\}$

from the steepest state we go first to the configuration $\{\ldots 11 \ldots 11 \ldots\}$ (the same but replacing the 0 and 2 by two 1's) with the thresholds equaling the slopes. So, any new added grain will travel the whole pile up to the exit. After this, for the site that has to have slope 2 we make its threshold 2; an additional incoming grain will make the slope at this site equal to 2 and that of the preceding site equal to zero; successive incoming grains will move the position of the zero to the desired site. When there are more zero-two pairs in the final configuration the generalization is straightforward if we start the previous procedure from the right.

This demonstrates that the condition that each zero must have a two to the right is a sufficient condition to belong to the attractor. But further, the previous reasoning shows that states violating this condition are not accessible from the steepest state, or from any other state which verifies the condition. So, the condition is necessary and sufficient, and the attractor proposed by Chua and Christensen constitutes the only final class of the system.

Finally, it is easy to show the existence of loops in the final class: any state without zeros can return to the same state after the addition of one grain if this grain travels through the whole pile and does not induce any other grain to topple. For this we need that sites with slope 1 have also thresholds equal to 1 and sites with slope 2 keep their thresholds equal to 2 after toppling. [The probability of this is $p^n q^{L-n}$, where n is the number of sites with $z(x)=1$.] This ensures the nonperiodicity of the graph and completes the demonstration of the existence of a unique stationary distribution of state occupation. In a case like this, the Markov chain is said to be regular.

Now that we know that there exists a stationary distribution, how do we obtain it? Notice that all that we have already learnt about the system has been accomplished without explicit knowledge of the transition probabilities $\mathcal{W}$; the relevant issue was if the transition was possible, $W_{ij}\neq 0$, or not. However, to calculate the stationary distribution and proceed further we need the calculation of the matrix elements.

IV. CALCULATION OF THE TRANSITION PROBABILITIES

It is possible to derive the transition probabilities between the metastable states in a pile of size L as a function of the transition probabilities for a size $L-1$. Since we have fully characterized the attractor [31], we restrict the calculation only to these states, for the sake of conciseness. Note that although the number of states in the attractor [Eq. (4)] becomes astronomically large for the usual values of L in the simulations, it constitutes a drastic reduction in front of the number of metastable states, which is 3^L; i.e.,

$$N_A \sim 2.6^L \ll 3^L, \tag{6}$$

for L large.

Starting from $L=1$ there are only two possible states in the attractor, $\{1\}$ and $\{2\}$ (state $\{0\}$ is clearly transient since the coordinate $x=1$ corresponds already to the boundary, where the toppling rules for the slope are special). We will

label these two states as 1 and 2 (in this case the label is straightforward, but not for larger L). Applying the rules of the model we easily get

$$\mathcal{W}^{(1)} = \begin{pmatrix} p & q \\ p & q \end{pmatrix}, \tag{7}$$

where the superindex (1) stresses that we are dealing with a system of size $L=1$.

We can now consider $L=2$ and look at the different states there, for instance, $\{11\}$. What happens when we add a grain at $x=1$? There is a probability p that the origin topples, if not, we end in a state $\{21\}$ with a probability q. If the origin topples, the grain jumps to position $x=2$ and there the problem reduces to an $L=1$ problem, for which we know the transition probabilities. In this case, we have to apply the transitions of state 1 (which remember are defined taking into account that one grain is added to this state at its leftmost position, $x=2$ now); as these transition probabilities are $W_{11}^{(1)}$ and $W_{12}^{(1)}$, we have for the state $\{11\}$ the probabilities

$$W_{\{11\}\{z(1)z(2)\}}^{(2)} = \begin{cases} q & \text{to go to state } \{z(1)z(2)\}=\{21\} \\ pW_{11}^{(1)}=p^2 & \text{to return to } \{11\} \\ pW_{12}^{(1)}=pq & \text{to go to } \{02\} \\ 0 & \text{for any other final state.} \end{cases} \tag{8}$$

This simple example shows how to reduce the problem from L to $L-1$. In general, we will refer to the L system as the pile, or just the system, and the $L-1$ pile, defined by $x=2,3,\ldots,L$, will be the subsystem or subpile. In the same way we will talk about states of the system and about substates when referring to the subsystem.

Although the previous case illustrates the basic idea, there appears an extra complication if the height at the origin, $h(1)$, is larger, which is that the origin can topple more than once if the avalanche in the $L-1$ subpile is big enough to leave the origin with a too large slope. In this case one can

apply the Abelian property: let us topple first the subpile (just using the transition probabilities $\mathcal{W}^{(L-1)}$ that we know) and then, at the end, let the origin topple. Of course, this gives rise to an iterative process, where the procedure has to be applied as many times as the origin topples, which is $h_I(1) - h_F(1) + 1$, I and F referring to the initial and final states. The sequence is: first the origin topples, then, the $L-1$ subpile to reach equilibrium, then the origin again (if needed), then the subpile, and so on.

Applying the previous argument to a general system of size L one can find the rules for the transition probabilities. It is convenient to define a variable Q as

$$Q \equiv h(1) - L = \sum_{x=1}^{L} z(x) - L; \tag{9}$$

we have $Q=0,1,\ldots,L$ in the attractor [31]. The equations for the elements of $\mathcal{W}^{(L)}$ will depend on $\Delta Q = Q_I - Q_F$, that is, the difference of Q between the initial state and the final state (defined here in the opposite way as usual); this is so because the number of topplings at $x=1$ is $\Delta Q+1$. The rules are given below and refer to the transition probabilities between an initial state I with a value of $z(1)=z_I$ and $L-1$ subpile state i and a final state F with $z(1)=z_F$ and substate f, that is, a transition $(z_I,i)\rightarrow(z_F,f)$. Note how an L state is completely characterized by the value of $z(1)$ and the state (substate) of the $L-1$ subsystem. As with every toppling of the origin Q decreases in one unit, we will use the following relation to calculate $z(1)$:

$$z(1) = Q - Q' + 1, \tag{10}$$

where Q' refers to the Q value of the $L-1$ subsystem. In general, Q'_s will denote the Q value of substate s in the $L-1$ subsystem. $T(z)$ will give the probability that a site topples for a given value of z [0, p, 1, for $z\leq 1$, 2, ≥ 3 respectively, see Eq. (5)]. Therefore, the argument of T is the value of $z(1)$ calculated after a number of topplings. With all these definitions the rules turn out to be

$$W_{(z_I,i)(z_F,f)}^{(L)} = \begin{cases} 0 & \text{if } \Delta Q \leq -2 \\ [1-T(z_I+1)]\delta_{if} & \text{if } \Delta Q = -1 \\ T(z_I+1)W_{if}^{(L-1)}[1-T(Q_I-Q'_f+1)] & \text{if } \Delta Q = 0 \\ T(z_I+1)\sum_j W_{ij}^{(L-1)}T(Q_I-Q'_j+1)W_{jf}^{(L-1)}[1-T(Q_I-Q'_f)] & \text{if } \Delta Q = 1 \\ T(z_I+1)\sum_j W_{ij}^{(L-1)}T(Q_I-Q'_j+1) \\ \quad \times \sum_k W_{jk}^{(L-1)}T(Q_I-Q'_k)W_{kf}^{(L-1)}[1-T(Q_I-Q'_f-1)] & \text{if } \Delta Q = 2. \end{cases} \tag{11}$$

The case $\Delta Q \leqslant -2$ is impossible since we would need adding more than one grain at $x=1$ (and that some of them would not topple) to reach the corresponding value of Q. For $\Delta Q = -1$ the only possibility is that the origin does not topple, then z_I increases in one unit and has to be stable, and the substate does not change. For the rest of cases we can write

$$W_{(z_I,i)(z_F,f)}^{(L)} = T(z_I+1) \sum_j W_{ij}^{(L-1)} T(Q_I-Q_j'+1)$$

$$\times \sum_k W_{jk}^{(L-1)} T(Q_I-Q_k')$$

$$\times \sum_l W_{kl}^{(L-1)} T(Q_I-Q_l'-1) \cdots$$

$$\times \sum_v W_{uv}^{(L-1)} T(Q_I-Q_v'-\Delta Q+2)$$

$$W_{vf}^{(L-1)}\{1-T[Q_I-Q_f'-(\Delta Q-1)]\} \quad \text{if} \quad \Delta Q \geqslant 0.$$
$$(12)$$

The general idea is that there is a probability $T(z_I+1)$ that the origin topples after the addition of one grain, then, there is a probability $W_{ij}^{(L-1)}$ to go to a substate j; with this substate there is a probability that the origin topples given by T

again, and then, one goes from j to k, from here to l, etc., following all the possible paths that end in f, whose origin has a probability $1-T(z_F)$ to be stable, with $z_F = Q_F - Q_f' + 1 = Q_I - Q_f' - \Delta Q + 1$.

The previous equations for the elements of $\mathcal{W}^{(L)}$ look like a matrix product but with different matrices and a different number of factors for different components. Nevertheless, in matrix form they can be written as

$$W_{(z_I,i)(z_F,f)}^{(L)} = T(z_I+1)[1-T(z_F)]$$

$$\times \left[\left(\prod_{Q=Q_I}^{Q_F+1} \mathcal{W}^{(L-1)}\mathcal{T}_Q\right)\mathcal{W}^{(L-1)}\right]_{if} \quad \text{if} \quad \Delta Q \geqslant 0,$$
$$(13)$$

where the index Q is assumed to decrease in the product, $[\ ]_{if}$ denotes that we take the element of the matrix in the i file and f row and $\mathcal{T}$ is a diagonal matrix whose elements can only be 1, p, or 0, more precisely,

$$[\mathcal{T}_Q]_{jk} = T(Q-Q_j'+1)\delta_{jk}.$$
$$(14)$$

We stress that the rules are valid for any pair of metastable states, although we will concentrate on states in the attractor.

These equations for $L=2$ yield

$$\mathcal{W}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ pq & p^2 & q & 0 & 0 \\ p^2q & p^3 & pq & q & 0 \\ p^3q & p^4 & p^2q & pq & q \\ p^3q(1+q) & p^4(1+q) & p^2q(1+q) & pq(1+q) & q^2 \end{pmatrix},$$
$$(15)$$

where states are ordered as $\{02\},\{11\},\{21\},\{12\},\{22\}$. Iterating the rules it is possible, although laborious, to generate the matrices $\mathcal{W}^{(L)}$ for successive L. Although the matrix for $L=2$ looks simple, the corresponding matrices as L increases are getting more and more complicated.

Figure 1 shows, for $L=7$ and 8, and $p=1/2$, the probability density $H(K)$ of the number of nonzero elements for each file of $\mathcal{W}$, that is, the distribution of the number of states directly accessible for a state in the attractor, or, in the language of networks, the out-degree distribution of the phase space. There seem to be two kinds of states, one group has few connections and the other one a large number of them; nevertheless, the system size is too small to be conclusive. In contrast, the in-degree distribution (not shown in the plot) looks rather uniform. (We will always use the letter H to denote probability densities, although it will correspond to different functions depending on the argument.)

In Fig. 2 we illustrate the enormous variation in the values of the transition probabilities: the probability density $H(W)$

that W_{ij} takes a certain value W is shown for $L=7$ and $p=1/2$, spanning about 16 orders of magnitude. A power law with exponent -1 approximates well this behavior. Curiously, a similar result has been found in networks describing correlations between earthquakes [38] and the solar coronal magnetic field [39] (with a different exponent in this case).

V. OBTAINING THE STATIONARY OCCUPATION DISTRIBUTION

To get the evolution of the system one has to start with a distribution of initial states P_0, which remember has to be an N_A-dimensional vector [see Eq. (4)], giving the occupation probability of any of the N_A states in the attractor. This initial distribution can be a delta [for instance, for $L=2$, starting always with $\{11\}$, i.e., $P_0 = (0,1,0,0,0)$] or not. The distribution of states after the first (slow) time step, i.e., after the addition of just one grain, is obtained as $P_1 = P_0\mathcal{W}$. To ob-
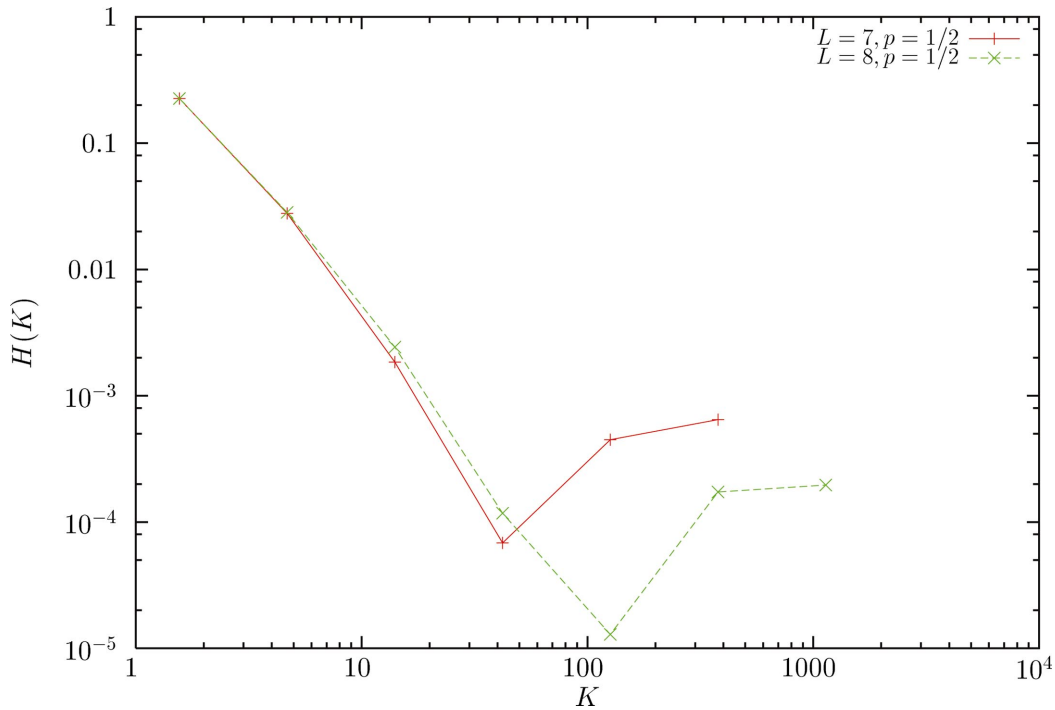
FIG. 1. (Color) Probability density that the number of states directly accessible in phase space (or out-degree distribution) for a state in the attractor takes a value equal to K, for $L=7$ and 8 and $p=1/2$. For K not very large, $H(K)$ could be a power law, but larger values of L are needed to be more sure. The histogram is calculated with exponentially increasing bins.

tain the distribution of states in the next time step we have to multiply again by $\mathcal{W}$ and so on; the powers of $\mathcal{W}$ give therefore the evolution of the system.

Let us call the vector representing the stationary distribution D, which of course is also a vector on N_A dimensions,

with each component $D(s)$ giving the probability that after a long enough time the system is in state s. The evolution of the occupation probabilities in the attractor is also obtained multiplying the row vector D by the matrix $\mathcal{W}$, but as D is the stationary distribution, it must be invariant under such
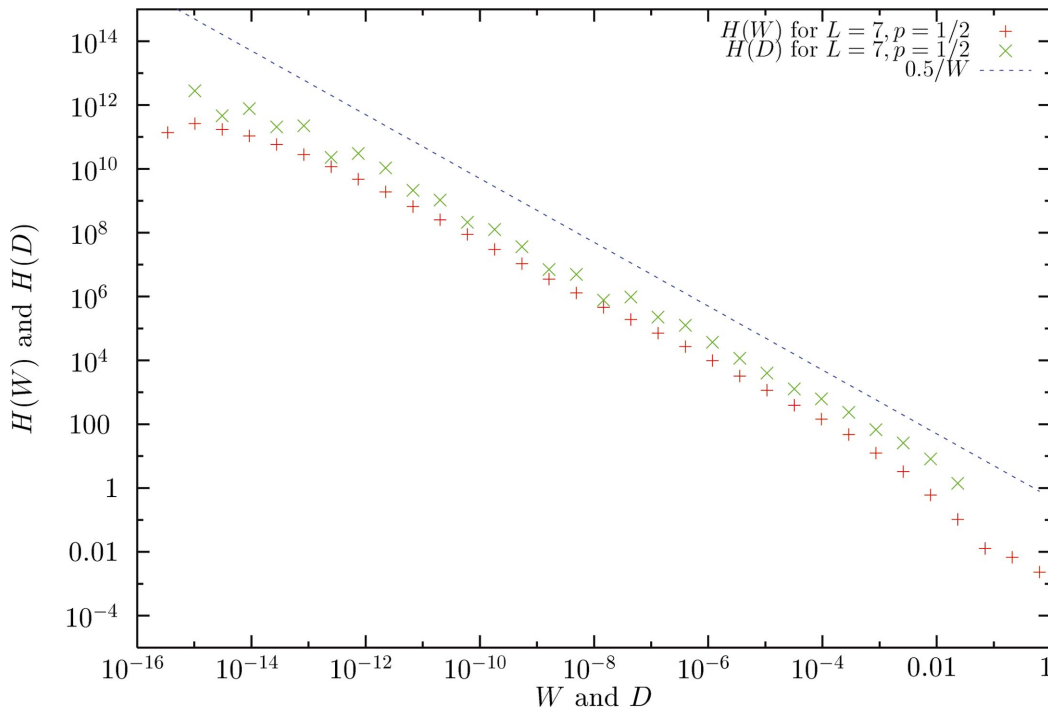

FIG. 2. (Color) Probability densities $H(W)$ and $H(D)$ that some transition probability takes a value W, and that some state s has stationary occupation probability $D(s)=D$, for $L=7$ and $p=1/2$.

operation, i.e., $D = D\mathcal{W}$, which simply means that D is a left eigenvector of $\mathcal{W}$ with eigenvalue equal to 1. Regularity (which was demonstrated in Sec. III) ensures that this eigenvector is unique.

Moreover, a direct consequence of regularity, which is also a sufficient condition for it, is that

$$\mathcal{M} \equiv \lim_{n \to \infty} \mathcal{W}^n = \begin{pmatrix} D \\ D \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ D \end{pmatrix}, \qquad (16)$$

which means that the transition matrix corresponding to n steps has all its files equal to the stationary distribution D, asymptotically. Indeed, it is trivial to show that for any distribution P, with $\Sigma_i P(i) = 1$, we have $P\mathcal{M} = D$. On the opposite side, if any P tends to D we can take $P = (1, 0 \ldots 0)$ to get the first file of matrix $\mathcal{M}$ (by multiplication), which must be equal to D, and the same can be done for any other file of $\mathcal{M}$.

If we consider the case $L = 2$ we realize that $\mathcal{M} = \mathcal{W}^3$, that is, we obtain a matrix whose files are all the same; this implies that the asymptotic result is reached in just three time steps, since $\mathcal{M} = \mathcal{W}\mathcal{M} = \mathcal{W}^2\mathcal{M}$, etc. But further, it turns out that the stationary distribution D, given by the files of $\mathcal{M}$, is the last file of $\mathcal{W}$, if the states are ordered by increasing Q, so,

$$D = [p^3 q(1+q), \ p^4(1+q), \ p^2 q(1+q),$$
$$pq(1+q), \ q^2]; \qquad (17)$$

that is, the transition probabilities of the steepest state $\{22\}$ give the occupation probabilities in the stationary regime. In other words, the unique eigenvector of $\mathcal{W}$ with eigenvalue equal to 1 is just its last file.

We have verified that this result is general for larger L, although the necessary number of powers increases with L (note that for $L = 1$ this is already accomplished at the first time step). Dhar [32] has beautifully demonstrated this result using the properties of an operator algebra. The idea behind this is simple: we can realize that it is equivalent to add $L(L+1)$ grains to the flattest state $\{00 \ldots 0\}$ (which is not in the attractor) than to add just one grain to the steepest state $\{22 \ldots 2\}$. Let us see why. The number of grains which separate both profiles is precisely $L(L+1)$, so, by application of the Abelian property, we add this number of grains to $\{00 \ldots 0\}$ and let them topple to reach the profile corresponding to the steepest state (the probability of this toppling process is exactly 1); after reaching this state we can continue the toppling process, but we are already in the same situation that results from adding just one grain to the steepest state. So we get the same configurations with the same probabilities in both cases. In fact this result is not only true for the flattest state, but for any other state, the only differ-

ence is that we will have some extra grains: no problem, they topple until they leave the pile. Therefore we can write

$$(0 \ldots 0, 1)\mathcal{W} = P_0 \mathcal{W}^{L(L+1)}, \qquad (18)$$

$\forall P_0 = (0 \ldots 0, 1, 0 \ldots 0)$. (This expresses something that we already proved in Sec. III, which is that the steepest state is reachable from any configuration, just adding enough grains.) Note now that the addition of one extra grain changes nothing in each case, this grain will topple until the exit, so

$$P_0 \mathcal{W}^{L(L+1)} = P_0 \mathcal{W}^{L(L+1)+1}. \qquad (19)$$

As this equality holds for any vector of the basis, we can write

$$\mathcal{W}^{L(L+1)} = \mathcal{W}^{L(L+1)+1} = \mathcal{M}, \qquad (20)$$

and so, from the first equation we get

$$(0 \ldots 0, 1)\mathcal{W} = P_0 \mathcal{M} = D, \quad \forall P_0, \qquad (21)$$

which means that, indeed, the transitions from the steepest state coincide with the stationary occupation distribution. Dhar has also noted that a more restrictive condition holds for the states in the attractor. The state there with less grains is $\{11 \ldots 1\}$; the difference in number of grains between the steepest state and this one is $L(L+1)/2$, which can replace the previous value $L(L+1)$, for any state in the attractor.

In Fig. 2 we also include the probability density $H(D)$ that $D(s)$ takes a given value for $L = 7$ and $p = 1/2$, showing a behavior very similar to the density of transition probabilities; this is a broad distribution across 13 orders of magnitude, close to a power law with exponent -1. This means that the occupation of the phase space (i.e., the space of all possible configurations) is enormously heterogeneous, at variance with the BTW model [34].

Figure 3 shows $D(s)$ as a function of s, where the states s are ordered in terms of decreasing $D(s)$. In fact, the form of $D(s)$ in this plot is related to $H(D)$, just by identifying s/N_A as the probability that the occupation probability is larger than (or equal to) a certain value $D(s)$, that is, as the survivor function of the random variable $D(s)$. Therefore, the density $H(D)$ will be (as usual) the derivative of this survivor function, multiplied by -1, or

$$H(D) = -\frac{1}{N_A}\left(\frac{dD(s)}{ds}\right)^{-1}. \qquad (22)$$

A power law with -1 exponent for $H(D)$ yields an exponentially decreasing $D(s)$, in agreement with the plot.

VI. CALCULATION OF THE DISTRIBUTIONS OF MEAN SLOPES AND AVALANCHE SIZES

From the values of the stationary distribution of the occupation of the states and their transition probabilities, $D(s)$ and $\mathcal{W}$, it is possible to calculate many things in the stationary state, (i.e., in the attractor). The first one is the (stationary) distribution of Q, $f(Q)$,
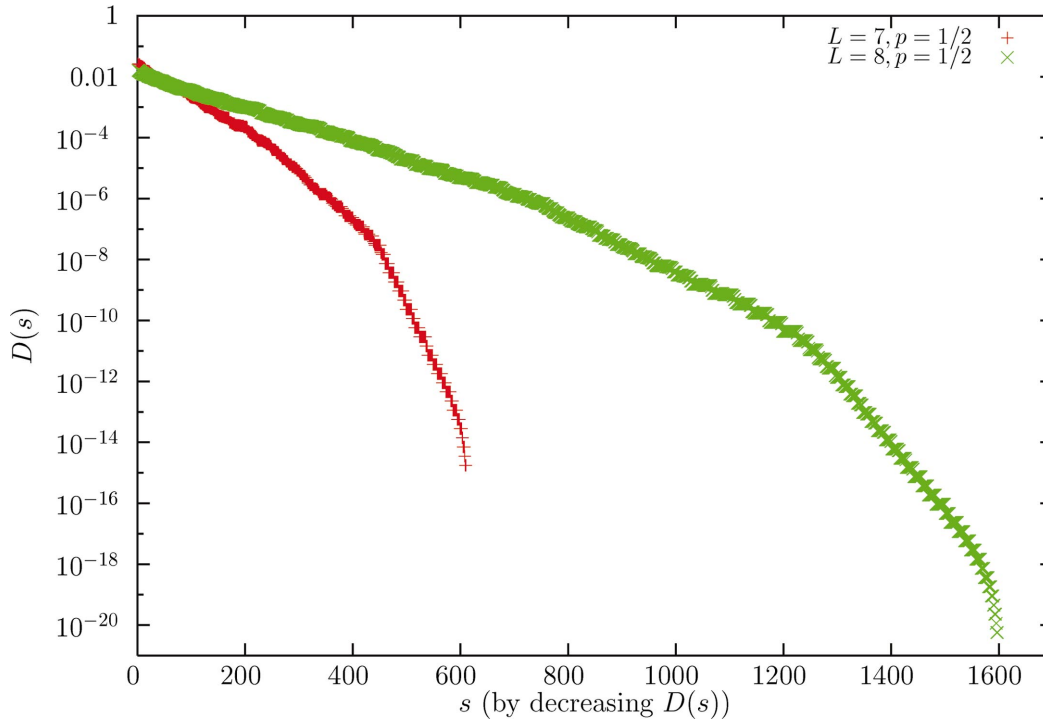
FIG. 3. (Color) Stationary probability of occupation $D(s)$ for each state for $L=7$ and 8, and $p=1/2$. The states are ordered in decreasing probability. Straight lines would indicate an exponential decay with s.

$$f(Q)=\sum_{\forall s s.t. Q_s=Q} D(s)=\sum_{\forall s} D(s)\delta_{Q_s,Q}. \qquad (23)$$

For example, for $L=2$ we get

$$f(Q)=[p^3(1+q), \quad pq(1+p)(1+q), \quad q^2], \qquad (24)$$

for $Q=0,1,2$.

The distribution $f(Q)$ is in fact the distribution of heights at the origin, $f(h(1))$, and it is also directly related to the distribution of mean slopes, $f(\bar{z})$, with $\bar{z}\equiv\Sigma_{x=1}^{L}z(x)/L$ $=h(1)/L$, since $Q=h(1)-L=L(\bar{z}-1)$. Consequently, $f(Q)$ defines the active zone width, which can be obtained as the standard deviation of this distribution. (For simplicity, we have used the same symbol f for all the distributions, although obviously they are not the same function.)

Figure 4 displays, for several values of L, the distribution of the value of $(Q-\langle Q\rangle)/L^{\chi}=[h(1)-\langle h(1)\rangle]/L^{\chi}=(\bar{z}-\langle\bar{z}\rangle)L^{1-\chi}$. For χ, we take the value proposed in Ref. [16], $\chi\simeq0.24$. Note how all the discrete distributions collapse under rescaling onto a single continuous curve, which is close to Gaussian, though slightly skewed: it is remarkable that scaling holds for the smallest system sizes. These exact results are in total agreement with the findings of computer simulations.

The avalanche size distribution $f(S)$ (in the attractor) is not difficult to calculate knowing $D(s)$ and $\mathcal{W}$. We can write

$$f(S)=\sum_{\forall s} p(S/s)D(s), \qquad (25)$$

where S is the avalanche size, s the state of the pile, and $p(S/s)$ is the conditional probability of having an avalanche of size S starting from a state s. For this term we have

$$p(S/s)=\sum_{\forall j s.t. S_{sj}=S} W_{sj}=\sum_{\forall j} W_{sj}\delta_{S_{sj},S}, \qquad (26)$$

where S_{sj} is the size of the avalanche triggered in the transition from s to j and can be calculated as

$$S_{sj}=\sum_{x=1}^{L} [h_s(x)-h_j(x)](L-x+1)+L, \qquad (27)$$

which is essentially the profile difference times the distance to the exit, plus the contribution of the added grain. Therefore we have

$$f(S)=\sum_{\forall s} D(s)\sum_{\forall j} W_{sj}\delta_{S_{sj},S}. \qquad (28)$$

The corresponding distribution calculated in this manner for $L=8$ and $p=1/2$ appears in Fig. 5, where it is compared with the result obtained from computer simulations. Note how even for such a small system there are avalanches with probability smaller than 10^{-22}.

VII. DISCUSSION

To summarize, we have shown that the conditions about the states put forward in Ref. [31] are necessary and sufficient to define the attractor of the Oslo model, which moreover is unique. In addition, its nonperiodical character allows
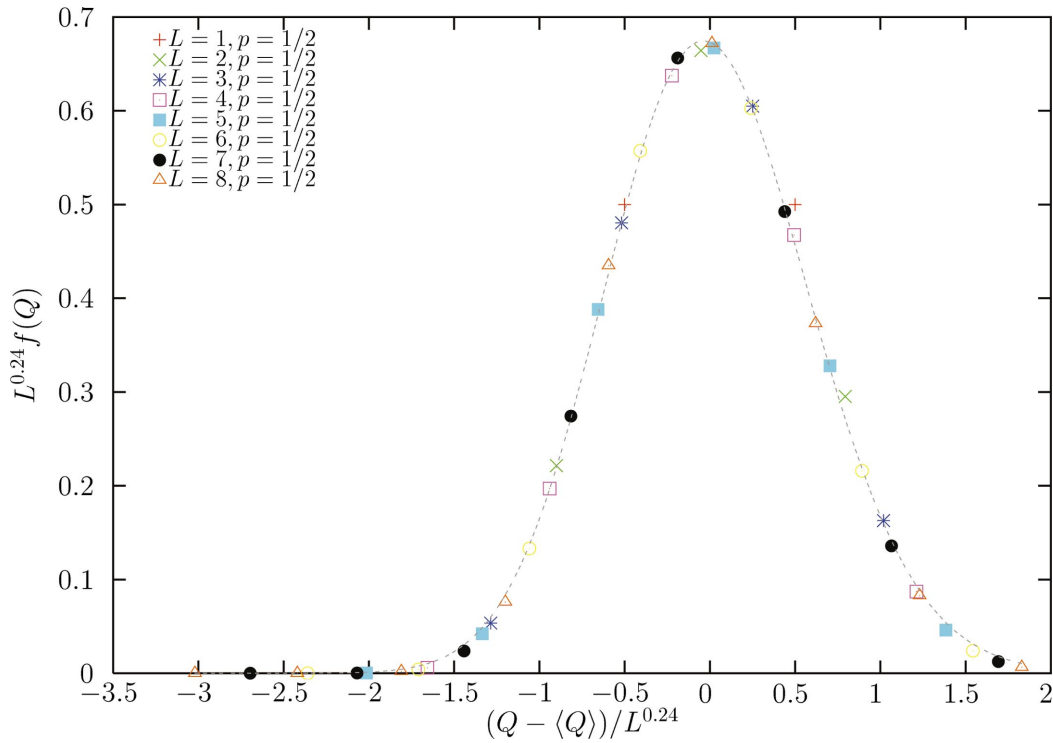
FIG. 4. (Color) Stationary distribution of Q [or of heights at the top, $h(1)$], centered by the mean and scaled by $L^{0.24}$. System size ranges from $L=1$ to $L=8$, and $p=1/2$. Notice how all the discrete distributions for each system size conspire to give a smooth curve, which is close to Gaussian, although slightly skewed.

the existence of a single occupation distribution in the stationary limit. The Abelian property enables the calculation of the transition probabilities between the different states, just from the decomposition of a system of size L into its leftmost column and an $L-1$ subsystem and by using an iterative toppling procedure; this allows to explore the network of connections in phase space. (Nevertheless, an in-depth investigation from the point of view of complex networks would
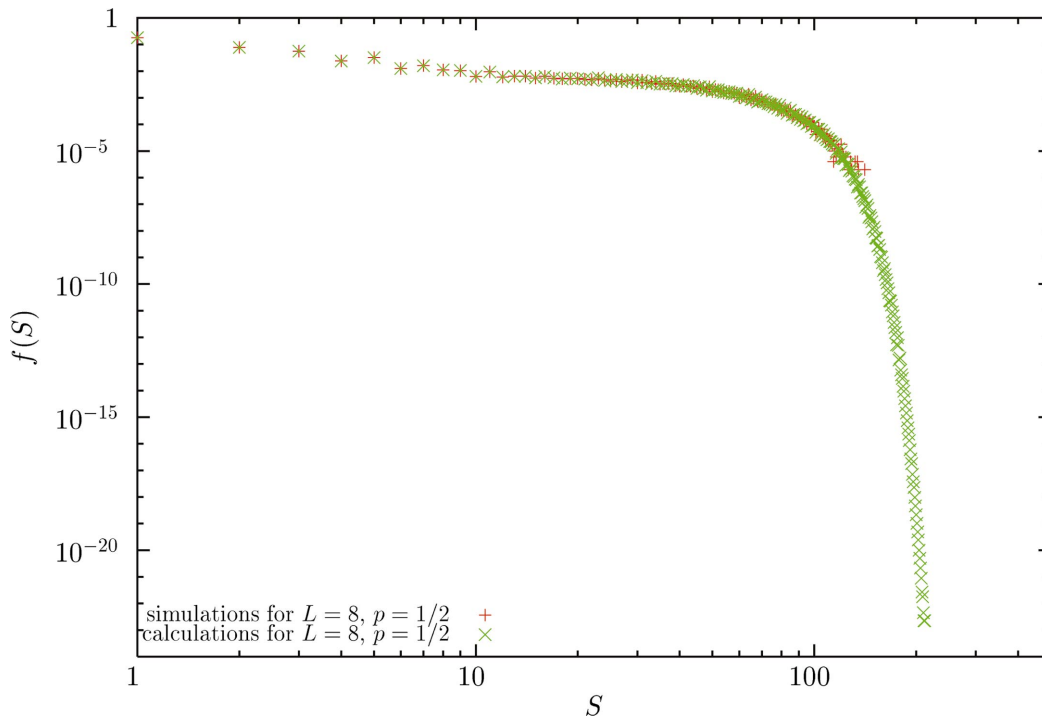

FIG. 5. (Color) Stationary distribution of avalanche sizes for $L=8$ and $p=1/2$ from our exact procedure and from simulations of the Oslo model.

be very illuminating, for sure [40].) Unexpectedly, the stationary occupation distribution turns out to be the last file of the transition matrix, i.e., that corresponding to the transitions of the steepest state. Both the transitions between states and their occupations can take values in an extremely large range of probabilities, setting a clear difference with other SOC systems. These calculations are exact for the system sizes involved, and could be performed for any L, in principle. In practice, we have strong limitations, as is explained below. Finally, with these quantities we can also derive the form of the fluctuations of the profile and the avalanche size distribution, for the corresponding value of L.

In fact, the knowledge of the transition probability matrix and the stationary distribution allows the calculation of any property related to the profile of the pile, as the ones we have just mentioned or the dissipated-energy distribution, the trapping-time distribution (approximately), and the damage spreading. But there are other properties that do not only depend on the profile but also on the dynamics which leads from one profile to another, for instance, the avalanche duration: knowing the profiles is not enough to calculate this quantity; in fact, with the same initial and final states the duration is not uniquely defined, i.e., there are many ways to end in the same state with a different number of avalanche time steps, depending on the actual sequence of topplings. So, despite the usefulness of the Abelian property, it does not allow to calculate dynamical magnitudes. This could be solved, in principle, with a parallel calculation stating the probability that a given transition would involve a given time; nevertheless, it would be rather complicated.

Also, the profiles, that is, the configurations defined in terms of slopes (or heights), do not retain any information regarding individual grains. Grains are therefore undistinguishable whereas for the calculation of transit times or flight lengths one needs distinguishable particles. This is another limitation of the current method difficult to overcome.

On the other hand, the equations obtained for the transition probabilities have the advantage that one does not have to simulate the system, and therefore one obtains exact probability distributions. The fact that for a small system of size $L=8$ there are occupation probabilities in the attractor smaller than 10^{-20}, and even smaller transition probabilities, indicates that the results we achieve would not have been possible just with computer simulations even for such small sizes. But the enormous number of states in the attractor (which increases as 2.6^L) makes impossible any calculation by hand beyond the smallest values of L. Symbolic computer calculations, performed by MAPLE or similar programs, have also severe limitations of size, and even with numeric computations the system size is limited to about $L \leq 10$ (for $L = 10$ there are more than 10^4 recurrent states, which would require 10^8 matrix elements). Also, the number of different matrix products gets very large with L. Although the results we obtain for small systems are interesting enough and representative of the complexity of the model, it would be nice to have access to supercomputers to increase the capabilities to manage larger system sizes.

To conclude, it is interesting to point out that having exact analytical results for some problem is not synonymous with understanding; in this case we can obtain exact expressions for the transition probabilities, the stationary distribution, or the avalanche size distribution, but from these formulas still it is not clear which are the properties of the system for large L. For instance, for the avalanche size distribution we can get very complicated exact equations, but we cannot show that they tend to a power law in the asymptotic limit. In any case, in my opinion, the results in this paper imply a remarkable progress in our picture of complex systems. Of course, the exact scheme presented here is amenable to application to extensions or variations of the Oslo model, and to other one-dimensional systems with boundary driving.

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